

Exponential decay phenomenon of the principal eigenvalue of an elliptic operator with a large drift term of gradient type

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Abstract

We study an asymptotic behaviour of the principal eigenvalue for an elliptic operator with large advection which is given by a gradient of a potential function. It is shown that the principal eigenvalue decays exponentially under the velocity potential well condition as the parameter tends to infinity. We reveal that the depth of the potential well plays an important role in the estimate. Particularly, in one dimensional case, we give a much more elaborate characterization for the eigenvalue. Some numerical examples obtained by a characteristic-curve finite element method are also shown.

1 Introduction

The following elliptic eigenvalue problem with a large drift term is considered in this paper. Let Ω be a bounded Lipschitz domain in \mathbb{R}^n ($n \geq 1$). For a given vector field $\mathbf{a} = (a_1, \dots, a_n)^T \in L^\infty(\Omega, \mathbb{R}^n)$, we consider an elliptic eigenvalue problem with a parameter $p \in \mathbb{R}$:

$$\begin{cases} -\Delta u(\mathbf{x}) + p \mathbf{a}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) = \lambda u(\mathbf{x}) & (\mathbf{x} \in \Omega) \\ u(\mathbf{x}) = 0 & (\mathbf{x} \in \partial\Omega) \end{cases} \quad (1.1)$$

We assume that the eigenfunction u ($u \not\equiv 0$) belongs to the Sobolev space $H_0^1(\Omega)$. Then, from the elliptic regularity theorem (see [9] for example), $u \in W_{\text{loc}}^{2,s}(\Omega)$ holds for arbitrary $s \in [1, \infty)$.

Although λ and $u(\mathbf{x})$ are complex-valued in general, we can define a positive real-valued principal eigenvalue $\lambda = \lambda_1(p)$ and the corresponding real-valued eigenfunction $u = u_1(\cdot, p)$, which is uniquely determined by the condition:

$$u(\mathbf{x}) > 0 \quad (\mathbf{x} \in \Omega), \quad \max_{\mathbf{x} \in \Omega} u(\mathbf{x}) = 1. \quad (1.2)$$

Some related results for the principal eigenvalue and eigenfunction will be collected in Theorem 3.1 and Theorem 3.7. A detailed and systematic study for the principal eigenvalue for general second order elliptic operators in general domains can be found in Berestycki, Nirenberg and Varadhan [3].

As we will see in Section 2 through several numerical examples, the principal eigenvalue is closely related to the decay rate of the solution of the corresponding nonstationary linear advection-diffusion equation (2.1).

Our interest in this study is asymptotic behaviours of $\lambda_1(p)$ as $p \rightarrow \infty$. Such large effects from advection or drift term under small diffusion appear in many actual physical problems and often cause some difficulties in their numerical simulations and analysis.

This phenomenon was first studied by Ventcel' [17] and Friedman [8] with Ventsel'-Freidlin's probabilistic approach [19]. In [8], the following result is obtained. If Ω is a bounded domain of C^2 -class, and if \mathbf{a} belongs to $C^1(\overline{\Omega})$ and satisfies the condition:

$$\mathbf{a}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) > 0 \quad (\mathbf{x} \in \partial\Omega), \quad (1.3)$$

where $\boldsymbol{\nu}$ denotes the outward unit normal vector on $\partial\Omega$, there exist c_1 and c_2 such that

$$0 < c_1 \leq \liminf_{p \rightarrow \infty} \frac{1}{p} \log \frac{1}{\lambda_1(p)} \leq \limsup_{p \rightarrow \infty} \frac{1}{p} \log \frac{1}{\lambda_1(p)} \leq c_2.$$

This estimate is equivalent to

$$\forall \varepsilon > 0, \quad \exists p_0 \in \mathbb{R} \quad \text{s.t.} \quad e^{-(c_2+\varepsilon)p} \leq \lambda_1(p) \leq e^{-(c_1-\varepsilon)p} \quad (\forall p \geq p_0),$$

which means that the principal eigenvalue becomes exponentially small under the condition (1.3).

In one dimensional case, asymptotic behaviours of k th eigenvalues $\lambda_k(p)$ ($k \in \mathbb{N}$) are studied in [4] by means of matched asymptotic expansions. In connection with a singular limit analysis of Sturm-Liouville two points boundary value problems, precise approximations of eigenvalues and eigenfunctions including exponentially small principal eigenvalue are obtained there.

For a given vector field $\mathbf{a} \in L^\infty(\Omega, \mathbb{R}^n)$, if the principal eigenvalue $\lambda_1(p)$ satisfies the condition:

$$\exists c > 0 \text{ and } \exists p_0 \in \mathbb{R} \quad \text{s.t.} \quad 0 < \lambda_1(p) \leq e^{-cp} \quad (\forall p \geq p_0), \quad (1.4)$$

or its equivalent condition:

$$\liminf_{p \rightarrow \infty} \frac{1}{p} \log \frac{1}{\lambda_1(p)} > 0,$$

then we call it exponential decay phenomenon of the principal eigenvalue. In addition to some numerical examples in Section 2, we will give a biological interpretation in a chemotaxis model for more physical pictures of the exponential decay phenomenon.

Besides the exponential decay phenomenon, the principal eigenvalue $\lambda_1(p)$ exhibits various asymptotic behaviour as $p \rightarrow \infty$. Devinatz, Ellis and Friedman [5] investigated L^2 inequalities type arguments and maximum principle type arguments, and they obtained some estimates of $\lambda_1(p)$ from above or from below by a term Cp^γ with $\gamma \in (0, 2]$. See also [18] for related results.

In [2], Berestycki, Hamel and Nadirashvili proved that, if $\partial\Omega$ is of C^2 -class and $\mathbf{a} \in L^\infty(\Omega, \mathbb{R}^n)$ satisfies $\text{div} \mathbf{a} = 0$ in $\mathcal{D}'(\Omega)$, then $\limsup_{p \rightarrow \infty} \lambda_1(p) < \infty$ if and only if there exists $\psi \in H^1(\Omega)$ such that $\psi \not\equiv \text{const}$ and $\mathbf{a} \cdot \nabla \psi = 0$ a.e. in Ω .

In our study, we focus on the exponential decay phenomenon and give some new estimates for it. We assume existence of a velocity potential of \mathbf{a} (condition (3.1)), which allows us to use the Rayleigh quotient and L^2 inequalities type arguments. One of our main results shows us that existence of a potential well (Definition 4.6) implies the exponential decay phenomenon of the principal eigenvalue, and that the depth of the potential well gives an estimate of the constant c in (1.4) from below (Theorem 4.7). This observation is extended to the exponential decay phenomenon of the m th eigenvalue (Theorem 4.8) and to a precise asymptotic behaviour of $\lambda_1(p)$ in one dimensional case (Theorem 5.2).

The organization of this paper is as follows. In Section 2, we show some numerical examples of the exponential decay phenomenon in two dimensional case. These numerical profiles of the principal eigenfunctions $u_1(\mathbf{x}, p)$ will be helpful in our analysis later. We collect some fundamental facts on the principal eigenvalue in Section 3. In Section 4, we prove several estimates

for asymptotic behaviours of $\lambda_1(p)$ including an alternative simpler proof of the exponential decay phenomenon. In Section 5, we give more precise asymptotic behaviour of $\lambda_1(p)$ in one dimensional case, under a different assumption than one of [4]. Additionally as an application, we give a biological interpretation of the exponential decay phenomena, life span of a biological colony under the chemotaxis effect in the last section.

2 Numerical examples of exponential decay phenomena

For the principal eigenvalues $\lambda_1(p)$ which is defined in (1.1), we give several examples of typical asymptotic behaviours of $\lambda_1(p)$ as $p \rightarrow \infty$ for fixed vector fields $\mathbf{a}(\mathbf{x})$, particularly focusing our interests on the exponential decay phenomenon. We start from the simplest case that \mathbf{a} is a constant vector field.

Example 2.1. Let $\mathbf{a} \in \mathbb{R}^n$ be a constant vector. Then we can easily check that

$$\lambda_1(p) = \lambda_1(0) + p^2 \frac{|\mathbf{a}|^2}{4}, \quad u_1(\mathbf{x}, p) = e^{\frac{p}{2} \mathbf{a} \cdot \mathbf{x}} u_1(\mathbf{x}, 0),$$

where $\lambda_1(0)$ and $u_1(\mathbf{x}, 0)$ are the principal eigenvalue of $-\Delta$ with the Dirichlet boundary condition on Ω and its corresponding eigenfunction. Moreover, not only the principal eigenvalue but also all eigenvalues are exactly shifted by $(|\mathbf{a}|^2/4)p^2$, i.e., $\lambda_k(p) = \lambda_k(0) + (|\mathbf{a}|^2/4)p^2$ for all $k \in \mathbb{N}$. In this case, the asymptotic behaviour of $\lambda_1(p)$ (or $\lambda_k(p)$) is of $O(p^2)$ as $p \rightarrow \infty$. We will see in the next section that this type of $O(p^2)$ behaviour is most common, for example if the vector field $\mathbf{a}(\mathbf{x})$ has no singular point. Of course, there is no chance to have the exponential decay phenomenon of principal eigenvalues as $p \rightarrow \infty$.

On the other hand, it seems to be difficult to give an example of the exponential decay phenomenon similarly only by using elementary function.

In this Section, we give some examples of the exponential decay phenomenon with the help of numerical simulations. For this purpose, we compute the following initial-boundary value problem of parabolic type:

$$\begin{cases} u_t - \Delta u + p \mathbf{a}(\mathbf{x}) \cdot \nabla u = 0 & (\mathbf{x} \in \Omega, 0 < t < \infty) \\ u(\mathbf{x}, t) = 0 & (\mathbf{x} \in \partial\Omega, t > 0) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & (\mathbf{x} \in \Omega), \end{cases} \quad (2.1)$$

where u_0 is a given positive initial function. In the following numerical examples, the domain is chosen as $\Omega = (-1, 1) \times (-1, 1)$. It is well-known that $u(\mathbf{x}, t)$ uniformly converges to zero as $t \rightarrow \infty$ and its asymptotic behaviour is more precisely given as

$$u(\mathbf{x}, t) \sim C e^{-\lambda_1(p)t} u_1(\mathbf{x}, p) \quad \text{as } t \rightarrow \infty, \quad (2.2)$$

where the constant C is positive if $u_0(\mathbf{x}) > 0$ for $\mathbf{x} \in \Omega$.

If $p \gg 1$, since (2.1) is related to exponentially small positive eigenvalue and is a convection-dominant problem, it requires us to adopt some upwinding technique for its reliable and reasonable numerical simulation. The following simulations were computed by means of a characteristic-curve finite element scheme (103) in Chap.3 of [14] with piecewise linear triangular elements. This is a upwind implicit scheme of first order based on the characteristic curve approximation:

$$u_t(\mathbf{x}, k\tau) + p \mathbf{a}(\mathbf{x}) \cdot \nabla u(\mathbf{x}, k\tau) = \frac{u(\mathbf{x}, k\tau) - u(\mathbf{x} - p \mathbf{a}(\mathbf{x})\tau, (k-1)\tau)}{\tau} + O(\tau),$$

where $\tau > 0$ is a small time increment. For more details and the stability analysis, see [13] and [14] etc.

We use the following vector field with compact support on \mathbb{R}^2 ;

$$\alpha(\mathbf{x}; R) := \begin{cases} \frac{\mathbf{x}}{|\mathbf{x}|} \sin\left(\frac{\pi|\mathbf{x}|}{R}\right) & (0 < |\mathbf{x}| \leq R) \\ \mathbf{0} & (\mathbf{x} = \mathbf{0}, \text{ or } |\mathbf{x}| > R) \end{cases}$$

We remark that $\alpha(\cdot; R) \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ and $\|\alpha(\cdot, R)\|_{L^\infty(\mathbb{R}^2)} = 1$.

Example 2.2. We set $\mathbf{a}(\mathbf{x}) := \alpha(\mathbf{x}; 1/2)$ and $u_0 \equiv 1$. For $p = 0, 10, 20, 30, 40$, we numerically computed (2.1) in the time interval $0 \leq t \leq 1.0$ and the results are shown in Figures 1–11. For each p , time evolution of $u(\mathbf{x}, t)$ and a profile of principal eigenfunction $u_1(\mathbf{x}, p)$ are shown in the figures.

These finite element simulations were computed with time increment $\tau = 1/2000$ on a unstructured triangular mesh of Ω which consists of 11292 total nodal points (degree of freedom including boundary points) and 22182 triangular elements with the average edge length 0.021. For the mesh generation, we used FreeFem++ [7].

Time evolutions of $u(\mathbf{x}, t)$ are drawn with fixed $x_2 = 0$, where the horizontal axis stands for $-1 \leq x_1 \leq 1$. In each simulation, for $t \geq 0.2$, $u(\mathbf{x}, t)$ decreases monotonically and keeps almost same profile, i.e., $u(\mathbf{x}, t)$ exhibits the asymptotical form (2.2). Comparing these figures, we can see that the decay speed of $u(\mathbf{x}, t)$ becomes extremely slow in cases of $p = 20, 30, 40$. Especially in case of $p = 40$ (Figure 9), it seems to be almost stationary after $t = 0.2$ in the figure. This is a typical example of the exponential decay phenomenon.

The principal eigenfunctions drawn in the figures are normalized profiles of $u(\mathbf{x}, t)$ at $t = 1.0$, which are considered as profiles of principal eigenfunctions $u_1(\mathbf{x}, p)$. The sections of normalized $u_1(\mathbf{x}, p)$ on the line $x_2 = 0$ are drawn in Figure 11. We remark that, in case of $p = 0$, the principal eigenvalue and eigenfunction are given by

$$\lambda_1(0) = \frac{\pi^2}{2}, \quad u_1(\mathbf{x}, 0) = \cos \frac{\pi x_1}{2} \cos \frac{\pi x_2}{2}.$$

Example 2.3. We set $\mathbf{x}_1 := (1/2, 2/5)$ and $\mathbf{x}_2 := (-2/3, -3/10)$ and define

$$\mathbf{a}(\mathbf{x}) := \alpha(\mathbf{x} - \mathbf{x}_1; 2/5) + 2\alpha(\mathbf{x} - \mathbf{x}_2; 1/4).$$

The support of the flow field consists of two disjoint sets $\{|\mathbf{x} - \mathbf{x}_1| \leq 2/5\}$ and $\{|\mathbf{x} - \mathbf{x}_2| \leq 1/4\}$.

For $p = 0, 10, 20, \dots, 100$, we numerically computed (2.1) in the time interval $0 \leq t \leq 1.0$ and the results are shown in Figures 12–22. For each $p = 10, 20, 30, 50, 100$, time evolution of $u(\mathbf{x}, t)$ and a profile of principal eigenfunction $u_1(\mathbf{x}, p)$ are shown in the figures. These finite element simulations were computed with same time increment and triangulation of Ω as Example 2.2.

Time evolutions of $u(\mathbf{x}, t)$ are drawn on the line $6x_1 - 10x_2 + 1 = 0$ through \mathbf{x}_1 and \mathbf{x}_2 , where the horizontal axis stands for $-1 \leq x_1 \leq 1$. Similarly to the previous example, this also exhibits the exponential decay phenomenon.

The principal eigenfunctions drawn in the figures are normalized profiles of $u(\mathbf{x}, t)$ at $t = 1.0$, which are considered as profiles of principal eigenfunctions $u_1(\mathbf{x}, p)$. The sections of normalized $u_1(\mathbf{x}, p)$ on the line $6x_1 - 10x_2 + 1 = 0$ are drawn in Figure 22.

From this and previous examples, the profiles of the eigenfunction $u_1(\mathbf{x}, p)$ seems to converge to a limit profile as $p \rightarrow \infty$. In particular, the limit profile seems to be flat on the support of \mathbf{a} in both cases.

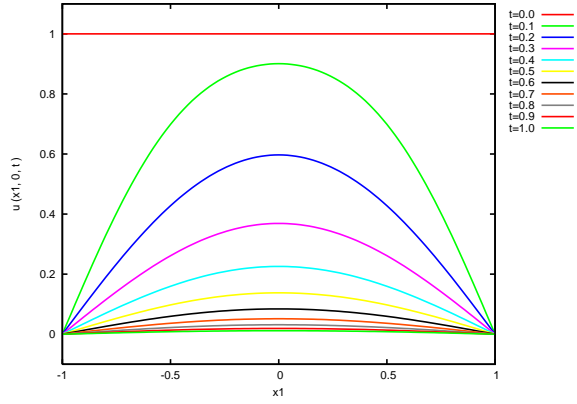


Figure 1: Time evolution of Example 2.2 on the line $x_2 = 0$ for $p = 0$.

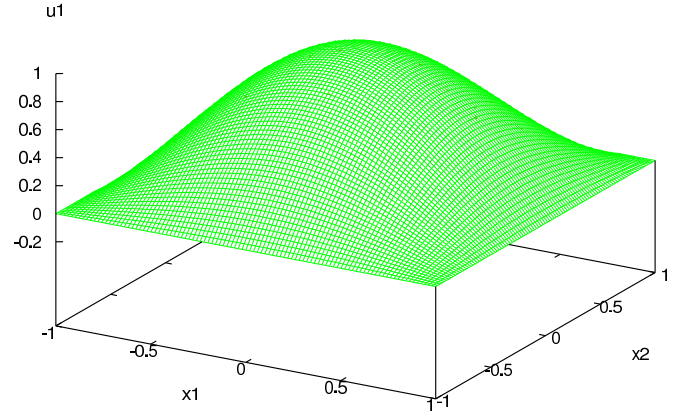


Figure 2: Eigenfunction of Example 2.2 for $p = 0$.

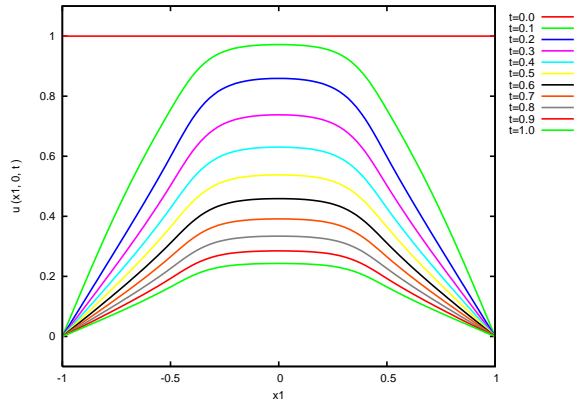


Figure 3: Time evolution of Example 2.2 on the line $x_2 = 0$ for $p = 10$.

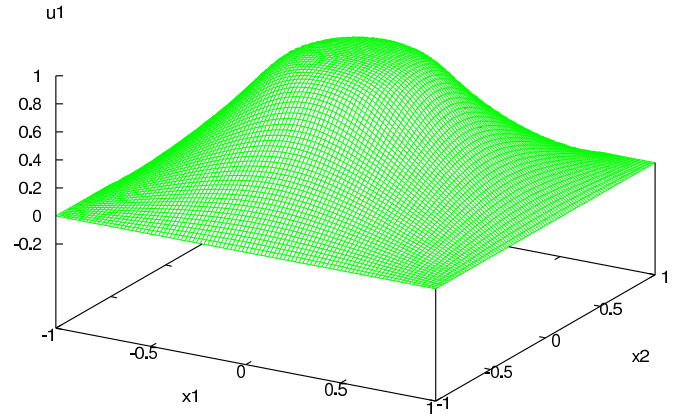


Figure 4: Eigenfunction of Example 2.2 for $p = 10$.

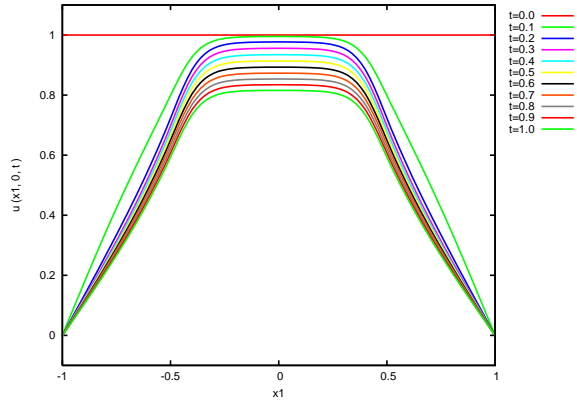


Figure 5: Time evolution of Example 2.2 on the line $x_2 = 0$ for $p = 20$.

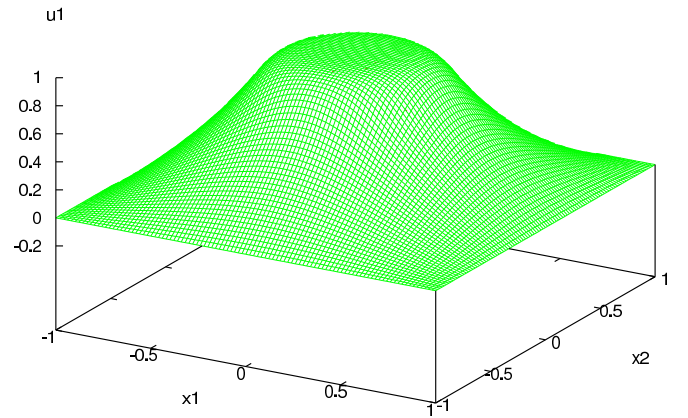


Figure 6: Eigenfunction of Example 2.2 for $p = 20$.

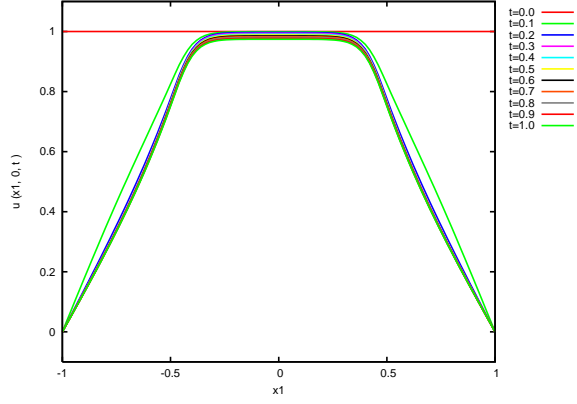


Figure 7: Time evolution of Example 2.2 on the line $x_2 = 0$ for $p = 30$.

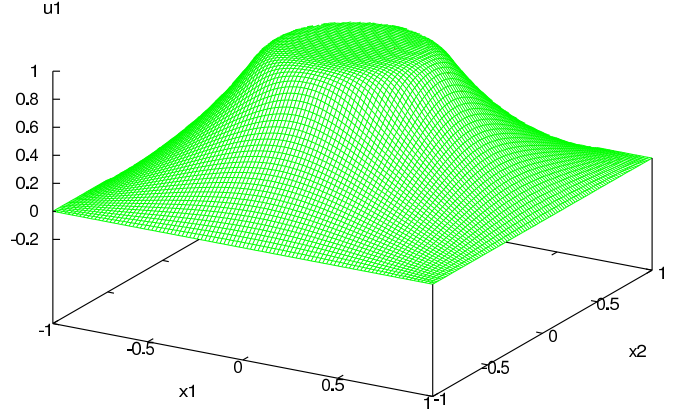


Figure 8: Eigenfunction of Example 2.2 for $p = 30$.

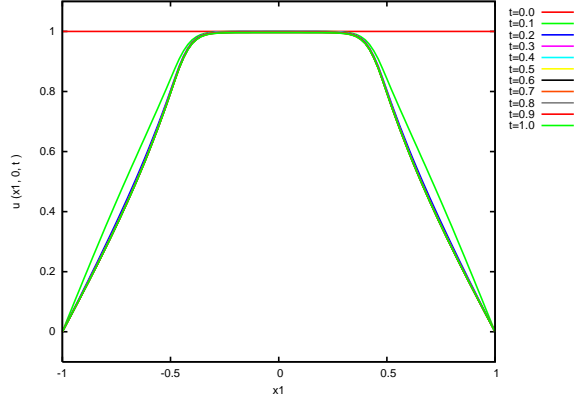


Figure 9: Time evolution of Example 2.2 on the line $x_2 = 0$ for $p = 40$.

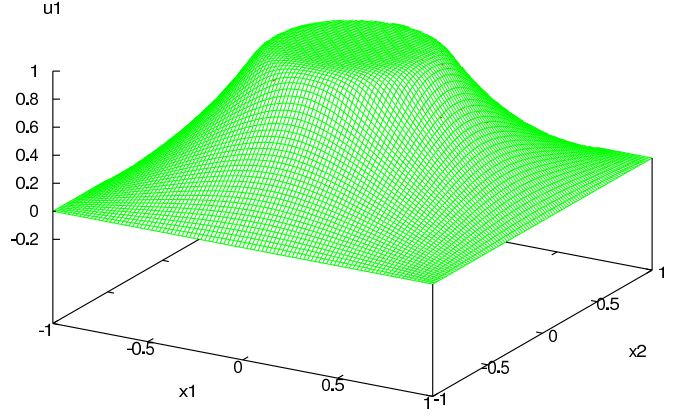


Figure 10: Eigenfunction of Example 2.2 for $p = 40$.

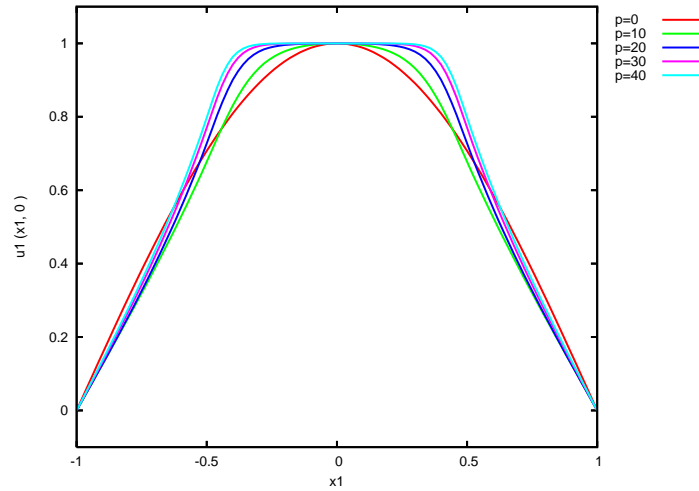


Figure 11: Eigenfunctions of Example 2.2 on the line $x_2 = 0$ for $p = 0, 10, 20, 30, 40$.

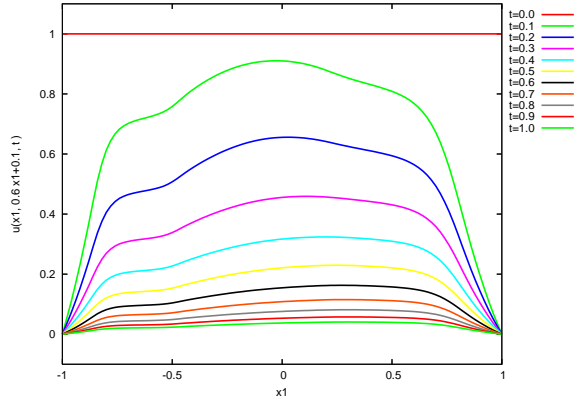


Figure 12: Time evolution of Example 2.3 on a line for $p = 10$.

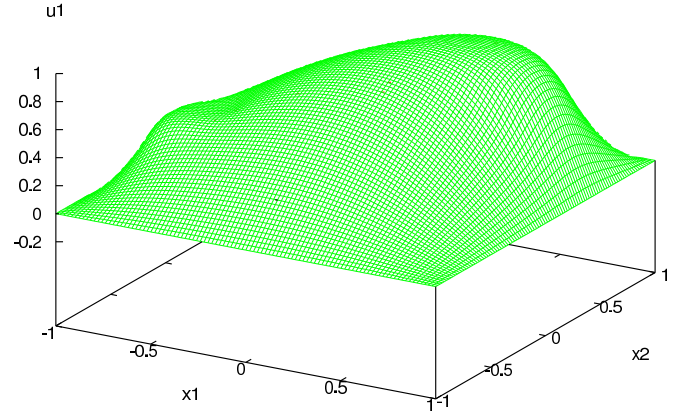


Figure 13: Eigenfunction of Example 2.3 for $p = 10$.

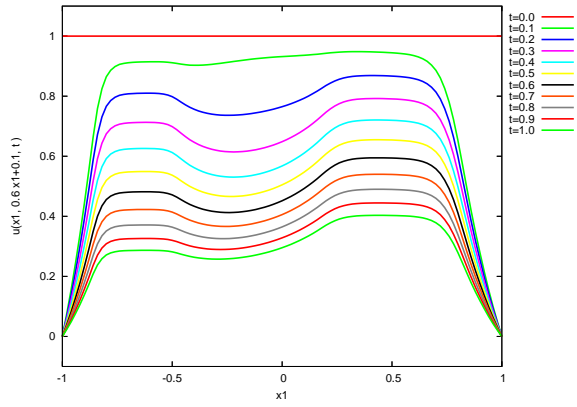


Figure 14: Time evolution of Example 2.3 on a line for $p = 20$.

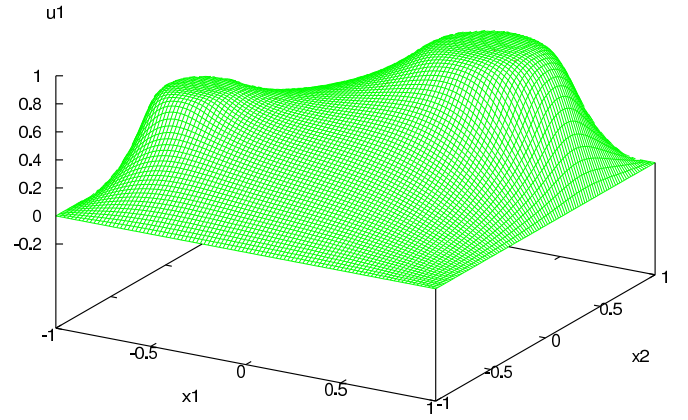


Figure 15: Eigenfunction of Example 2.3 for $p = 20$.

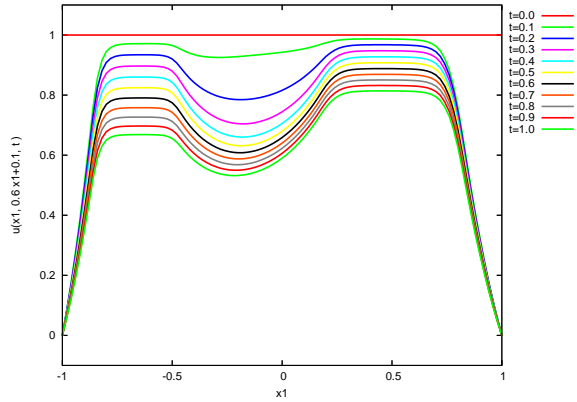


Figure 16: Time evolution of Example 2.3 on a line for $p = 30$.

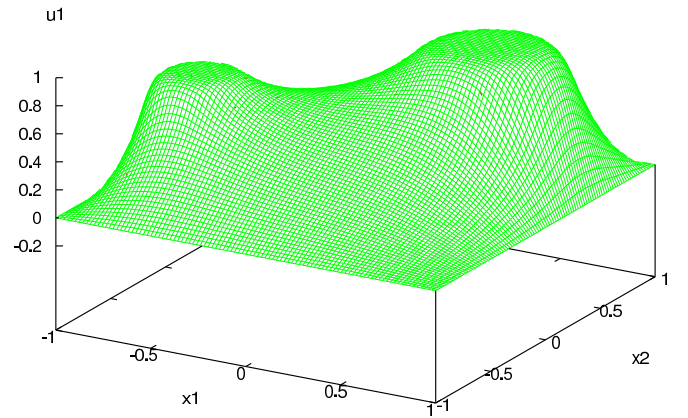


Figure 17: Eigenfunction of Example 2.3 for $p = 30$.

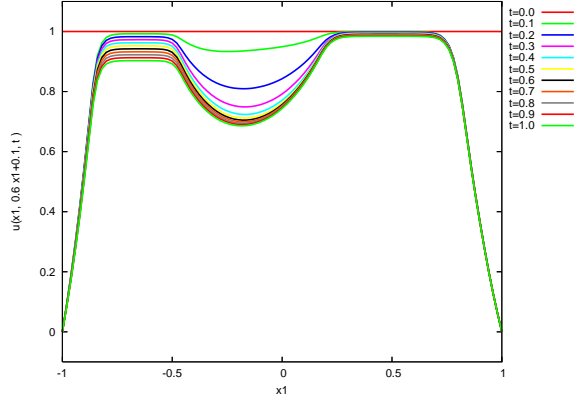


Figure 18: Time evolution of Example 2.3 on a line for $p = 50$.

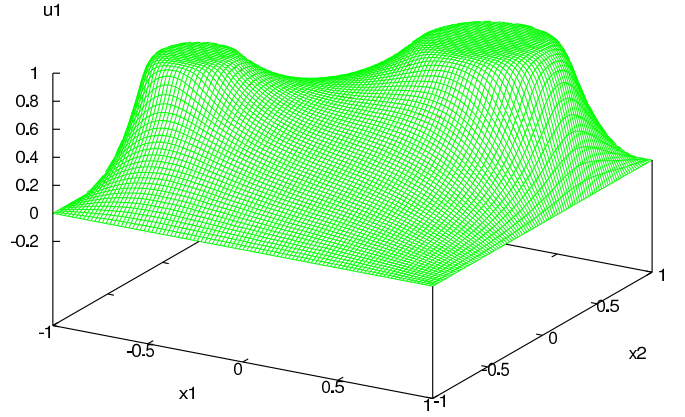


Figure 19: Eigenfunction of Example 2.3 for $p = 50$.

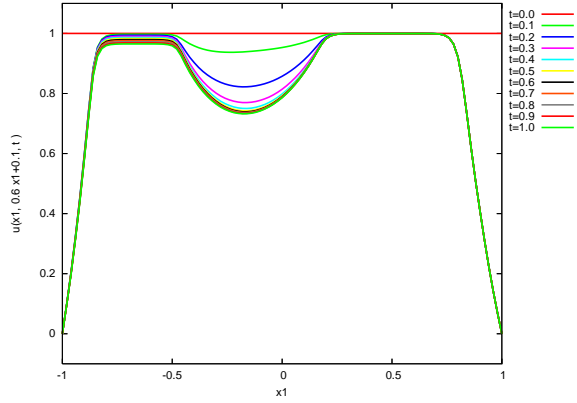


Figure 20: Time evolution of Example 2.3 on a line for $p = 100$.

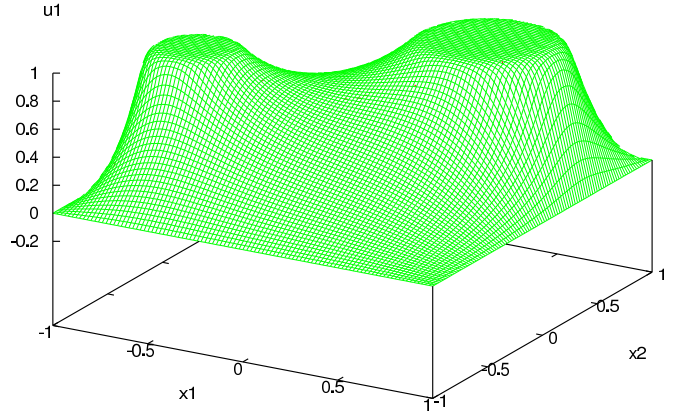


Figure 21: Eigenfunction of Example 2.3 for $p = 100$.

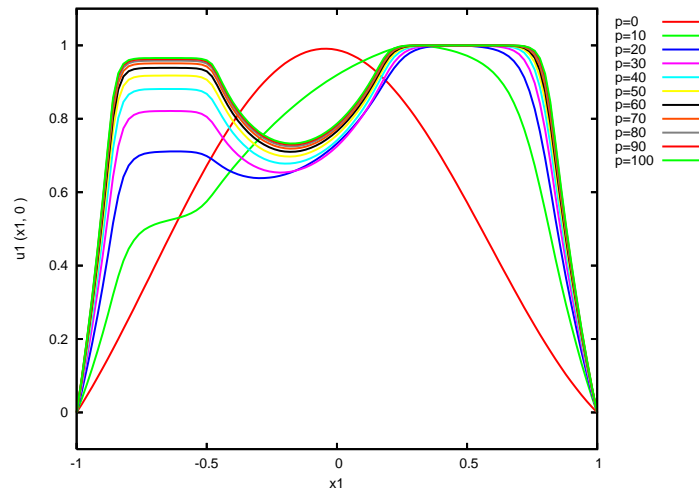


Figure 22: Eigenfunctions of Example 2.3 on a line for $p = 0, 10, 20, \dots, 100$.

3 Fundamental tools for eigenvalue problems

In this section, we collect several fundamental facts and tools for the eigenvalue problem. For simple notation, we fix $p = 1$ without loss of generality throughout this section.

Theorem 3.1. *In a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$), we assume $\mathbf{a} \in L^\infty(\Omega, \mathbb{R}^n)$. Then, there uniquely exists a positive number λ_1 and $u_1 \in H_{\text{loc}}^2(\Omega) \cap H_0^1(\Omega)$ such that $\lambda = \lambda_1$ and $u = u_1$ satisfy (1.1) with $p = 1$ and (1.2). Moreover, λ_1 is given by the min-max formula:*

$$\lambda_1 = \max_{\varphi > 0} \inf_{\mathbf{x} \in \Omega} \left(\frac{-\Delta \varphi(\mathbf{x}) + \mathbf{a}(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x})}{\varphi(\mathbf{x})} \right),$$

where $\max_{\varphi > 0}$ is taken over all positive $\varphi \in W_{\text{loc}}^{2,n}(\Omega)$.

We remark that inf and sup always denote essential infimum and supremum in this paper. For the proof of the theorem, see Section 2.8 of [15] and [3]. Since $\lambda_1 = \min_\lambda \text{Re} \lambda$ holds for any complex eigenvalue λ of $-\Delta + \mathbf{a} \cdot \nabla$, λ_1 and u_1 are called the principal eigenvalue and eigenfunction. The next corollary immediately follows from Theorem 3.1.

Corollary 3.2. *Under the condition of Theorem 3.1, we consider a Lipschitz subdomain $\Omega' \subset \Omega$ and define $\lambda'_1 > 0$ as the principal eigenvalue for $-\Delta u + \mathbf{a} \cdot \nabla u = \lambda'_1 u$ in Ω' with the zero Dirichlet boundary condition on $\partial\Omega'$. Then $\lambda_1 \leq \lambda'_1$ holds.*

Throughout the following sections, we assume the existence of a velocity potential of \mathbf{a} :

$$\exists b \in W^{2,\infty}(\Omega) \text{ s.t. } \mathbf{a}(\mathbf{x}) = \nabla b(\mathbf{x}) \quad (\mathbf{x} \in \Omega). \quad (3.1)$$

Since the existence of a velocity potential is not essential for the exponential decay phenomena which is our aim in this paper (see [8], for example), the assumption (3.1) is rather technical but allows us energy-based arguments with the help of the following Liouville transform.

Under the condition (3.1) and for $q \in L^\infty(\Omega)$, we consider the following three elliptic eigenvalue problems with the Dirichlet boundary condition.

$$\begin{cases} -\Delta u(\mathbf{x}) + \mathbf{a}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) = \lambda u(\mathbf{x}) & (\mathbf{x} \in \Omega) \\ u(\mathbf{x}) = 0 & (\mathbf{x} \in \partial\Omega) \end{cases} \quad (3.2)$$

$$\begin{cases} -\Delta v(\mathbf{x}) - \text{div}(v(\mathbf{x})\mathbf{a}(\mathbf{x})) = \lambda v(\mathbf{x}) & (\mathbf{x} \in \Omega) \\ v(\mathbf{x}) = 0 & (\mathbf{x} \in \partial\Omega) \end{cases} \quad (3.3)$$

$$\begin{cases} -\Delta w(\mathbf{x}) + q(\mathbf{x})w(\mathbf{x}) = \lambda w(\mathbf{x}) & (\mathbf{x} \in \Omega) \\ w(\mathbf{x}) = 0 & (\mathbf{x} \in \partial\Omega) \end{cases} \quad (3.4)$$

Proposition 3.3 (Liouville transform). *Suppose the condition (3.1). Then the above three eigenvalue problems are equivalent to each other under the following relations:*

$$e^{\frac{b}{2}} v = e^{-\frac{b}{2}} u = w, \quad q = -\frac{1}{2} \Delta b + \frac{1}{4} |\nabla b|^2.$$

We omit the proof, since it is shown by straightforward substitution. We remark that the same Liouville transform is valid even for the time dependent problem (2.1).

For the self-adjoint eigenvalue problem (3.4) with $q \in L^\infty(\Omega)$, there is a well-known min-max principle for the Rayleigh quotient. We define

$$L_q := -\Delta + q, \quad \text{Dom}(L_q) = H_0^1(\Omega), \quad (3.5)$$

$$J_q(w) := \frac{H^{-1}(\Omega) \langle L_q w, w \rangle_{H_0^1(\Omega)}}{(w, w)_{L^2(\Omega)}} = \frac{\int_\Omega (|\nabla w(\mathbf{x})|^2 + q(\mathbf{x})w(\mathbf{x})^2) d\mathbf{x}}{\int_\Omega w(\mathbf{x})^2 d\mathbf{x}} \quad (w \in H_0^1(\Omega)).$$

Theorem 3.4. *For $q \in L^\infty(\Omega)$, there exists a complete orthonormal system of $L^2(\Omega)$, $\{w_k\}_{k \in \mathbb{N}} \subset H_0^1(\Omega) \cap H_{\text{loc}}^2(\Omega)$ and $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ ($\lambda_1 < \lambda_2 \leq \dots$) such that*

$$L_q w_k = \lambda_k w_k \quad (k \in \mathbb{N}).$$

The k th eigenvalue λ_k is characterized by the min-max formula:

$$\lambda_k = \min_{\dim X=k} \max_{w \in X \setminus \{0\}} J_q(w),$$

where the minimum is taken over whole k -dimensional subspace X in $H_0^1(\Omega)$. In particular, the principal eigenvalue λ_1 is simple and given by

$$\lambda_1 = \min_{w \in H_0^1(\Omega) \setminus \{0\}} J_q(w).$$

Moreover, an eigenfunction $w \in H_0^1(\Omega) \cap H_{\text{loc}}^2(\Omega)$ corresponds to the principal eigenvalue if and only if w has no sign change in Ω .

For the proof of this theorem, see Section 8.12 of [9] and [1] etc.

From the last assertion of this theorem, without loss of generality, we can assume that

$$w_1(\mathbf{x}) > 0 \quad (\mathbf{x} \in \Omega).$$

Theorem 3.5 (comparison theorem for principal eigenvalues). *For $q, \tilde{q} \in L^\infty(\Omega)$, the principal eigenvalues of L_q and $L_{\tilde{q}}$ are denoted by λ_1 and $\tilde{\lambda}_1$, respectively. Then*

$$\inf_\Omega (q - \tilde{q}) \leq \lambda_1 - \tilde{\lambda}_1 \leq \sup_\Omega (q - \tilde{q}).$$

In particular, if $q(\mathbf{x}) \geq \tilde{q}(\mathbf{x})$ ($\mathbf{x} \in \Omega$) and $\|q - \tilde{q}\|_{L^\infty(\Omega)} \neq 0$, then $\lambda_1 > \tilde{\lambda}_1$.

Proof. Let w_1 and \tilde{w}_1 denote the eigenfunctions corresponding to λ_1 and $\tilde{\lambda}_1$, respectively. We have

$$\begin{aligned} \lambda_1 &= \min_{w \in H_0^1(\Omega), w \neq 0} J_q(w) \leq J_q(\tilde{w}_1) = J_{\tilde{q}}(\tilde{w}_1) + (J_q(\tilde{w}_1) - J_{\tilde{q}}(\tilde{w}_1)) \\ &= \tilde{\lambda}_1 + \frac{\int_\Omega (q(\mathbf{x}) - \tilde{q}(\mathbf{x})) \tilde{w}_1(\mathbf{x})^2 d\mathbf{x}}{\int_\Omega \tilde{w}_1(\mathbf{x})^2 d\mathbf{x}} \leq \tilde{\lambda}_1 + \sup_\Omega (q - \tilde{q}). \end{aligned}$$

In the same way, we also have

$$\tilde{\lambda}_1 \leq \lambda_1 + \sup_\Omega (\tilde{q} - q) = \lambda_1 - \inf_\Omega (q - \tilde{q}). \quad (3.6)$$

These inequalities imply the first assertion. Furthermore, if $q \geq \tilde{q}$ and $q \not\equiv \tilde{q}$ then, from the positivity of w_1 and \tilde{w}_1 , we have

$$\lambda_1(w_1, \tilde{w}_1)_{L^2(\Omega)} = (L_q w_1, \tilde{w}_1)_{L^2(\Omega)} = (w_1, L_q \tilde{w}_1)_{L^2(\Omega)} > (w_1, L_{\tilde{q}} \tilde{w}_1)_{L^2(\Omega)} = \tilde{\lambda}_1(w_1, \tilde{w}_1)_{L^2(\Omega)},$$

and $\lambda_1 > \tilde{\lambda}_1$ follows. □

We define the elliptic operator K_b by

$$K_b := -\Delta + \nabla b \cdot \nabla, \quad \text{Dom}(K_b) = H_0^1(\Omega),$$

and denote by K_b^* its adjoint operator with respect to $L^2(\Omega)$:

$$K_b^* := -\Delta - \text{div}(\cdot \nabla b), \quad \text{Dom}(K_b^*) = H_0^1(\Omega).$$

Although the following results concerning the operator K_b^* will not be used in our analysis, we include them below for our systematic description. Another reason to include them is that the eigenvalue problem (3.3) corresponding to K_b^* is related to various important applications as well as (3.2) for K_b . An example of such applications will be shown in Section 6.

We introduce the following weighted inner product of $L^2(\Omega)$;

$$(v, u)_{L_b^2(\Omega)} := \int_{\Omega} e^{b(\mathbf{x})} v(\mathbf{x}) u(\mathbf{x}) d\mathbf{x}, \quad (v, u \in L^2(\Omega)),$$

and we define $L_b^2(\Omega) := (L^2(\Omega), (\cdot, \cdot)_{L_b^2(\Omega)})$ which denotes a Hilbert space $L^2(\Omega)$ with this inner product. We also define

$$I_b(u) := \frac{\int_{\Omega} e^{-b(\mathbf{x})} |\nabla u(\mathbf{x})|^2 d\mathbf{x}}{\int_{\Omega} e^{-b(\mathbf{x})} u(\mathbf{x})^2 d\mathbf{x}} \quad (u \in H_0^1(\Omega), u \neq 0),$$

$$I_b^*(v) := \frac{\int_{\Omega} e^{b(\mathbf{x})} \{|\nabla v(\mathbf{x})|^2 - \Delta b(\mathbf{x}) v(\mathbf{x})^2\} d\mathbf{x}}{\int_{\Omega} e^{b(\mathbf{x})} v(\mathbf{x})^2 d\mathbf{x}} \quad (v \in H_0^1(\Omega), v \neq 0).$$

Then we have the following proposition.

Proposition 3.6. *We suppose that $u, v, \varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ ($u, v \neq 0$). Then we have*

$$(K_b u, \varphi)_{L_{-b}^2(\Omega)} = (u, K_b \varphi)_{L_{-b}^2(\Omega)}, \quad I_b(u) = \frac{(K_b u, u)_{L_{-b}^2(\Omega)}}{(u, u)_{L_{-b}^2(\Omega)}},$$

$$(K_b^* v, \varphi)_{L_b^2(\Omega)} = (v, K_b^* \varphi)_{L_b^2(\Omega)}, \quad I_b^*(v) = \frac{(K_b^* v, v)_{L_b^2(\Omega)}}{(v, v)_{L_b^2(\Omega)}}.$$

Namely, K_b and K_b^* are selfadjoint in $L_{-b}^2(\Omega)$ and $L_b^2(\Omega)$, respectively, and their Rayleigh quotients are given by $I_b(u)$ and $I_b^*(v)$.

Proof. For $\varphi = u$ or v , by the integration by parts, we obtain

$$\int_{\Omega} e^{\mp b} |\nabla \varphi|^2 d\mathbf{x} = - \int_{\Omega} \text{div} \left(e^{\mp b} \nabla \varphi \right) \varphi d\mathbf{x} = \int_{\Omega} e^{\mp b} (-\Delta \varphi \pm \nabla b \cdot \nabla \varphi) \varphi d\mathbf{x}.$$

Hence, we have

$$I_b(u) = \frac{\int_{\Omega} e^{-b} (-\Delta u + \nabla b \cdot \nabla u) u d\mathbf{x}}{(u, u)_{L_{-b}^2(\Omega)}} = \frac{\int_{\Omega} e^{-b} (K_b u) u d\mathbf{x}}{(u, u)_{L_{-b}^2(\Omega)}} = \frac{(K_b u, u)_{L_{-b}^2(\Omega)}}{(u, u)_{L_{-b}^2(\Omega)}},$$

$$I_b^*(v) = \frac{\int_{\Omega} e^b \{(-\Delta v - \nabla b \cdot \nabla v) v - (\Delta b) v^2\} d\mathbf{x}}{(v, v)_{L_b^2(\Omega)}} = \frac{\int_{\Omega} e^b (K_b^* v) v d\mathbf{x}}{(v, v)_{L_b^2(\Omega)}} = \frac{(K_b^* v, v)_{L_b^2(\Omega)}}{(v, v)_{L_b^2(\Omega)}}.$$

The self-adjointness of K_b in $L_{-b}^2(\Omega)$ and the one of K_b^* in $L_b^2(\Omega)$ are also checked by direct calculations. \square

We have the following characterizations of the eigenvalues of (3.2) and (3.3).

Theorem 3.7. *For $b \in W^{2,\infty}(\Omega)$, there exist $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ ($\lambda_1 < \lambda_2 \leq \dots$), and $\{u_k\}_{k \in \mathbb{N}}, \{v_k\}_{k \in \mathbb{N}} \subset H_0^1(\Omega) \cap H_{\text{loc}}^2(\Omega)$ such that*

$$K_b u_k = \lambda_k u_k, \quad K_b^* v_k = \lambda_k v_k \quad (k \in \mathbb{N}),$$

and that $\{u_k\}_{k \in \mathbb{N}}$ and $\{v_k\}_{k \in \mathbb{N}}$ are complete orthonormal systems of $L_{-b}^2(\Omega)$ and $L_b^2(\Omega)$, respectively. The k th common eigenvalue λ_k is characterized by the min-max formula:

$$\lambda_k = \min_{\dim X=k} \max_{u \in X \setminus \{0\}} I_b(u) = \min_{\dim X=k} \max_{v \in X \setminus \{0\}} I_b^*(v),$$

where the minimum is taken over whole k -dimensional subspace X in $H_0^1(\Omega)$. In particular, the common principal eigenvalue λ_1 is simple and given by

$$\lambda_1 = \min_{u \in H_0^1(\Omega) \setminus \{0\}} I_b(u) = \min_{v \in H_0^1(\Omega) \setminus \{0\}} I_b^*(v).$$

Moreover, an eigenfunction u for K_b (v for K_b^) corresponds to the principal eigenvalue if and only if u (v) has no sign change in Ω .*

Proof. The assertions follow from Proposition 3.3 and Theorem 3.4. \square

From the last assertion of this theorem, without loss of generality, we can assume that

$$u_1(\mathbf{x}) > 0, \quad v_1(\mathbf{x}) > 0 \quad (\mathbf{x} \in \Omega).$$

As a last remark in this section, we introduce the following proposition. We also omit the proof since it is shown by a direct calculation.

Proposition 3.8. *Let $\Omega := \prod_{i=1}^n \mathcal{I}_i \subset \mathbb{R}^n$, where \mathcal{I}_i is a bounded open interval for $i = 1, \dots, n$, and let $\mathbf{a}(\mathbf{x}) = (a_1(x_1), a_2(x_2), \dots, a_n(x_n))^T$ for $\mathbf{x} = (x_1, \dots, x_n)^T \in \Omega$ with $a_i \in L^\infty(\mathcal{I}_i)$. Then the principal eigenvalue λ_1 and the eigenfunction $u_1(\mathbf{x})$ of (3.2) are given by the following formula:*

$$\lambda_1 = \sum_{i=1}^n \lambda_1^{(i)}, \quad u_1(\mathbf{x}) := \prod_{i=1}^n \varphi_i(x_i) \quad (\mathbf{x} \in \Omega),$$

where $\lambda_1^{(i)}$ and φ_i are defined by

$$\begin{cases} -\varphi_i''(x) + a_i(x)\varphi_i'(x) = \lambda_1^{(i)} \varphi_i(x) & (x \in I_i) \\ \varphi_i(x) = 0 & (x \in \partial I_i) \\ \varphi_i(x) > 0 & (x \in I_i). \end{cases}$$

4 Asymptotic behaviour of principal eigenvalues

In this section, we consider the singular perturbation problem of the principal eigenvalues (1.1) under the velocity potential condition (3.1). Henceforth we assume that p is a positive parameter.

Due to Proposition 3.3, (1.1) is equivalent to each of the following three eigenvalue problems under the zero Dirichlet boundary condition:

$$K_{pb}u = -\Delta u + p\nabla b \cdot \nabla u = \lambda u \quad \text{in } \Omega,$$

$$K_{pb}^*v = -\Delta v - p \operatorname{div}(v\nabla b) = \lambda v \quad \text{in } \Omega,$$

$$L_{q(p)}w = -\Delta w + q(p)w = \lambda w \quad \text{in } \Omega,$$

where we define

$$q(\mathbf{x}, p) := -\frac{p}{2} \operatorname{div} \mathbf{a}(\mathbf{x}) + \frac{p^2}{4} |\mathbf{a}(\mathbf{x})|^2, \quad (4.1)$$

and we abbreviate $q(\cdot, p) (\in L^\infty(\Omega))$ as $q(p)$.

Under the condition (3.1), the k th eigenvalue (their multiplicities are counted) of these equivalent eigenvalue problems is denoted by $\lambda_k(p)$ for a parameter $p > 0$.

We start from the following simple consequence of the comparison theorem.

Theorem 4.1. *Let $\lambda_\Omega > 0$ be the principal eigenvalue of $-\Delta_D$, which is Laplacian with the zero Dirichlet boundary condition. Under the condition (3.1), following two estimates hold:*

$$\lambda_1(p) \geq \lambda_\Omega - \frac{p}{2} \sup_{\mathbf{x} \in \Omega} (\operatorname{div} \mathbf{a}(\mathbf{x})) \quad (p > 0).$$

$$\frac{1}{4} \inf_{\mathbf{x} \in \Omega} |\mathbf{a}(\mathbf{x})|^2 \leq \liminf_{p \rightarrow \infty} \frac{\lambda_1(p)}{p^2} \leq \limsup_{p \rightarrow \infty} \frac{\lambda_1(p)}{p^2} \leq \frac{1}{4} \sup_{\mathbf{x} \in \Omega} |\mathbf{a}(\mathbf{x})|^2.$$

Proof. Applying Theorem 3.5 to $L_{q(p)}$ and $L_0 = -\Delta_D$, we have

$$\inf_{\mathbf{x} \in \Omega} q(\mathbf{x}, p) \leq \lambda_1(p) - \lambda_\Omega \leq \sup_{\mathbf{x} \in \Omega} q(\mathbf{x}, p), \quad (4.2)$$

and

$$\lambda_1(p) \geq \lambda_\Omega + \inf \left(-\frac{p}{2} \operatorname{div} \mathbf{a} + \frac{p^2}{4} |\mathbf{a}|^2 \right) \geq \lambda_\Omega - \frac{p}{2} \sup (\operatorname{div} \mathbf{a}) + \frac{p^2}{4} \inf |\mathbf{a}|^2, \quad (4.3)$$

$$\lambda_1(p) \leq \lambda_\Omega + \sup \left(-\frac{p}{2} \operatorname{div} \mathbf{a} + \frac{p^2}{4} |\mathbf{a}|^2 \right) \leq \lambda_\Omega - \frac{p}{2} \inf (\operatorname{div} \mathbf{a}) + \frac{p^2}{4} \sup |\mathbf{a}|^2. \quad (4.4)$$

The first assertion of the theorem follows from (4.3). Dividing (4.3) and (4.4) by p^2 and taking the limit $p \rightarrow \infty$, we can also derive the second assertion. \square

Corollary 4.2. *There is no exponential decay phenomenon if $\inf |\mathbf{a}(\mathbf{x})| > 0$ or $\sup \operatorname{div} \mathbf{a}(\mathbf{x}) \leq 0$.*

The estimate (4.2) can be improved as follows.

Lemma 4.3. *For $p_1 > p_2 \geq 0$, we have*

$$\inf_{\mathbf{x} \in \Omega} q(\mathbf{x}, p_1 + p_2) \leq \frac{p_1 + p_2}{p_1 - p_2} (\lambda_1(p_1) - \lambda_1(p_2)) \leq \sup_{\mathbf{x} \in \Omega} q(\mathbf{x}, p_1 + p_2). \quad (4.5)$$

Proof. From the definition of $q(\mathbf{x}, p)$ (4.1), the equality

$$q(\mathbf{x}, p_1) - q(\mathbf{x}, p_2) = \frac{p_1 - p_2}{p_1 + p_2} q(\mathbf{x}, p_1 + p_2),$$

holds. Hence, the assertion follows from Theorem 3.5. \square

From (4.2) and (4.5), it is natural to consider the following condition for \mathbf{a} :

$$\exists p_0 > 0 \text{ s.t. } \inf_{\mathbf{x} \in \Omega} q(\mathbf{x}, p_0) \geq 0. \quad (4.6)$$

Since

$$\frac{q(\mathbf{x}, p)}{p} = -\frac{1}{2} \operatorname{div} \mathbf{a} + \frac{p}{4} |\mathbf{a}|^2,$$

the inequality

$$\frac{q(\mathbf{x}, p_1)}{p_1} \geq \frac{q(\mathbf{x}, p_2)}{p_2} \quad (p_1 \geq p_2 > 0), \quad (4.7)$$

holds. Hence, the condition (4.6) implies $\inf_{\mathbf{x} \in \Omega} q(\mathbf{x}, p) \geq 0$ for all $p \geq p_0$. From Lemma 4.3, we have the following theorem.

Theorem 4.4. *Suppose the condition (4.6). Then, $\lambda_1(p)$ is nondecreasing for $p \geq p_0/2$. In particular, the exponential decay phenomenon does not occur in this case.*

Proof. We assume that $p_1 > p_2 \geq p_0/2$. Then we have $p_1 + p_2 > p_0$. From the first inequality of Lemma 4.3 and (4.7), we obtain

$$\lambda_1(p_1) - \lambda_1(p_2) \geq \frac{p_1 - p_2}{p_1 + p_2} \inf q(p_1 + p_2) = (p_1 - p_2) \inf \frac{q(p_1 + p_2)}{p_1 + p_2} \geq (p_1 - p_2) \inf \frac{q(p_0)}{p_0} \geq 0.$$

\square

We notice the following necessary condition for (4.6).

Proposition 4.5. *The condition (4.6) implies that*

$$\min_{\mathbf{x} \in \overline{\Omega'}} b(\mathbf{x}) = \min_{\mathbf{x} \in \partial\Omega'} b(\mathbf{x}) \quad (\Omega': \text{an arbitrary subdomain of } \Omega). \quad (4.8)$$

Proof. We remark that the condition (4.6) is equivalent to $\mathcal{K}b \geq 0$, where \mathcal{K} is a linear differential operator defined by

$$\mathcal{K}u := -\Delta u + \frac{p_0}{2} \nabla b \cdot \nabla u.$$

Since b is a supersolution with respect to \mathcal{K} , from a consequence of the weak maximum principle ([9], Theorem 3.1), the condition (4.8) follows. \square

An inverse condition of (4.8) is given by the following potential well condition.

Definition 4.6. *For a fixed velocity potential $b \in W^{2,\infty}(\Omega)$, a Lipschitz (nonempty) subdomain $\Omega' \subset \Omega$ is called a potential well if the condition*

$$\min_{\mathbf{x} \in \overline{\Omega'}} b(\mathbf{x}) < \min_{\mathbf{x} \in \partial\Omega'} b(\mathbf{x}),$$

is satisfied. Furthermore,

$$b_0 := \min_{\mathbf{x} \in \partial\Omega'} b(\mathbf{x}) - \min_{\mathbf{x} \in \overline{\Omega'}} b(\mathbf{x}) > 0.$$

is called the depth of a potential well Ω' .

We remark that if there exists a potential well, the condition (4.8) does not hold, and neither does (4.6). Moreover, according to the next theorem, existence of a potential well implies the exponential decay phenomenon of principal eigenvalues.

Let Ω' be a subdomain of C^2 -class and let $\boldsymbol{\nu}$ denote the outward unit normal vector on $\partial\Omega'$. If it satisfies the condition:

$$\mathbf{a}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) > 0 \quad (\mathbf{x} \in \partial\Omega'), \quad (4.9)$$

which is a sufficient condition for the exponential decay phenomenon in Ω' obtained in [8], then the exponential decay phenomenon occurs also in Ω due to Corollary 3.2. Actually, under the condition (3.1), the condition (4.9) implies that Ω' is a potential well.

Theorem 4.7. *We suppose that there exists a potential well Ω' with depth $b_0 > 0$. Then an exponential decay phenomenon occurs:*

$$\liminf_{p \rightarrow \infty} \frac{1}{p} \log \frac{1}{\lambda_1(p)} \geq b_0.$$

In other words, for any $\omega \in (0, b_0)$, there exists $C > 0$ such that

$$0 < \lambda_1(p) \leq C e^{-\omega p} \quad (p \geq 0).$$

Proof. Without loss of generality, we can assume $\min_{\mathbf{x} \in \overline{\Omega'}} b(\mathbf{x}) = 0$. From Theorem 3.7, $\lambda_1(p)$ is given by

$$\lambda_1(p) = \min_{u \in H_0^1(\Omega), u \neq 0} I_{pb}(u).$$

Hence, for arbitrary $u \in H_0^1(\Omega)$ ($u \neq 0$) and $\beta \in \mathbb{R}$, we have

$$\lambda_1(p) \leq I_{pb}(u) = \frac{\int_{\Omega} e^{-p(b-\beta)} |\nabla u|^2 d\mathbf{x}}{\int_{\Omega} e^{-p(b-\beta)} u^2 d\mathbf{x}}. \quad (4.10)$$

Let us choose β satisfying $0 < \beta < \beta + \omega < b_0$. Since $b \in C^0(\overline{\Omega})$ and $\min_{\partial\Omega'} b = b_0 > \beta + \omega$, there exists $\varepsilon > 0$ such that $b(\mathbf{x}) \geq \beta + \omega$ for all $\mathbf{x} \in N^\varepsilon(\partial\Omega')$, where

$$N^\varepsilon(\partial\Omega') := \{\mathbf{x} \in \Omega'; \text{dist}(\mathbf{x}, \partial\Omega') < \varepsilon\}, \quad \text{dist}(\mathbf{x}, \partial\Omega') := \min\{|\mathbf{x} - \mathbf{y}|; \mathbf{y} \in \partial\Omega'\}.$$

We define $\hat{u} \in H_0^1(\Omega) \cap C^{0,1}(\overline{\Omega})$ by

$$\hat{u}(\mathbf{x}) := \begin{cases} 0 & (\mathbf{x} \in \overline{\Omega} \setminus \Omega') \\ \varepsilon^{-1} \text{dist}(\mathbf{x}, \partial\Omega') & (\mathbf{x} \in N^\varepsilon(\partial\Omega')) \\ 1 & (\mathbf{x} \in \Omega' \setminus N^\varepsilon(\partial\Omega')), \end{cases} \quad (4.11)$$

and substitute it to (4.10). Since $|\nabla \hat{u}| = \varepsilon^{-1}$ a.e. in $N^\varepsilon(\partial\Omega')$ and $|\nabla \hat{u}| = 0$ in $\Omega \setminus \overline{N^\varepsilon(\partial\Omega')}$, we obtain

$$\lambda_1(p) \leq \frac{\int_{N^\varepsilon(\partial\Omega')} e^{-p(b(\mathbf{x})-\beta)} \varepsilon^{-2} d\mathbf{x}}{\int_{\Omega' \setminus N^\varepsilon(\partial\Omega')} e^{-p(b(\mathbf{x})-\beta)} d\mathbf{x}} \leq \frac{|N^\varepsilon(\partial\Omega')| \varepsilon^{-2} e^{-\omega p}}{|\{\mathbf{x} \in \Omega'; b(\mathbf{x}) \leq \beta\}|} = C e^{-\omega p},$$

where

$$C := \varepsilon^{-2} |\{\mathbf{x} \in \Omega'; b(\mathbf{x}) \leq \beta\}|^{-1} |N^\varepsilon(\partial\Omega')|.$$

□

The above theorem can be extended to first m eigenvalues $\lambda_1(p), \dots, \lambda_m(p)$, if there are m potential wells.

Theorem 4.8. *Let $\Omega_j \subset \Omega$ ($j = 1, \dots, m$) be disjoint potential wells ($\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset$ for $i \neq j$), and let ω_0 be the minimum depth of the potential wells Ω_j ($j = 1, \dots, m$). Then, for any $\omega \in (0, \omega_0)$, there exists $C > 0$ such that*

$$0 < \lambda_1(p) < \lambda_2(p) \leq \dots \leq \lambda_m(p) \leq Ce^{-\omega p} \quad (p \geq 0).$$

Proof. For each $j = 1, \dots, m$, we define $\hat{u}_j \in H_0^1(\Omega) \cap C^{0,1}(\overline{\Omega})$ with $\text{supp}(\hat{u}_j) = \overline{\Omega_j}$ in the same manner as (4.11). Let X be a linear subspace of $H_0^1(\Omega)$ generated by functions $\hat{u}_1, \dots, \hat{u}_m$. It is easy to see $\dim X = m$. Due to the min-max formula in Theorem 3.7, the m th eigenvalue $\lambda_m(p)$ is estimated from above as

$$\lambda_m(p) \leq \max_{u \in X \setminus \{0\}} I_{pb}(u) = \max_{c_j} I_{pb}(c_1 \hat{u}_1 + \dots + c_m \hat{u}_m), \quad (4.12)$$

where in \max_{c_j} , the maximum is taken over all $(c_1, \dots, c_m) \in \mathbb{R}^m \setminus \{(0, \dots, 0)\}$. Since $\overline{\Omega_j}$ are disjoint, we have

$$\begin{aligned} I_{pb}(c_1 \hat{u}_1 + \dots + c_m \hat{u}_m) &= \frac{\int_{\Omega} e^{-pb(\mathbf{x})} \left(\sum_{j=1}^m c_j^2 |\nabla \hat{u}_j(\mathbf{x})|^2 \right) d\mathbf{x}}{\int_{\Omega} e^{-pb(\mathbf{x})} \left(\sum_{j=1}^m c_j^2 |\hat{u}_j(\mathbf{x})|^2 \right) d\mathbf{x}} = \frac{\sum_{j=1}^m c_j^2 \int_{\Omega_j} e^{-pb(\mathbf{x})} |\nabla \hat{u}_j(\mathbf{x})|^2 d\mathbf{x}}{\sum_{j=1}^m c_j^2 \int_{\Omega_j} e^{-pb(\mathbf{x})} |\hat{u}_j(\mathbf{x})|^2 d\mathbf{x}} \\ &= \frac{\sum_{j=1}^m d_j^2 \frac{\int_{\Omega_j} e^{-pb(\mathbf{x})} |\nabla \hat{u}_j(\mathbf{x})|^2 d\mathbf{x}}{\int_{\Omega_j} e^{-pb(\mathbf{x})} |\hat{u}_j(\mathbf{x})|^2 d\mathbf{x}}}{\sum_{j=1}^m d_j^2} \leq \max_{1 \leq j \leq m} \frac{\int_{\Omega_j} e^{-pb(\mathbf{x})} |\nabla \hat{u}_j(\mathbf{x})|^2 d\mathbf{x}}{\int_{\Omega_j} e^{-pb(\mathbf{x})} |\hat{u}_j(\mathbf{x})|^2 d\mathbf{x}}, \end{aligned}$$

where we have defined $d_j := c_j (\int_{\Omega_j} e^{-pb(\mathbf{x})} |\hat{u}_j(\mathbf{x})|^2 d\mathbf{x})^{1/2}$.

Similar to the proof of Theorem 4.7, choosing sufficiently small $\varepsilon > 0$, we can show that the last term is bounded by $Ce^{-\omega p}$ from above. \square

5 Precise asymptotic behaviour in one dimensional case

In this section, we introduce a more precise estimate for the exponential decay phenomenon of the principal eigenvalues in one dimensional case. In [12], some of results in this section have been already announced in Japanese by one of the authors.

We consider the following one dimensional eigenvalue problem on $\Omega = (-l, l)$, where $l > 0$.

$$\begin{cases} -u''(x) + pa(x)u'(x) = \lambda u(x) & (-l < x < l) \\ u(-l) = u(l) = 0 \\ u(x) > 0 & (-l < x < l). \end{cases} \quad (5.1)$$

The principal eigenvalue is denoted by $\lambda_1(p) > 0$. In the following argument, we fix one of the primitive functions of $a \in L^\infty(-l, l)$ and denote it by $b \in W^{1,\infty}(-l, l) = C^{0,1}([-l, l])$, i.e.,

$$b'(x) = a(x) \quad \text{a.e. } x \in (-l, l).$$

We also define

$$b_0 := b_2 - b_1, \quad b_1 := \min_{0 \leq x \leq l} b(x), \quad b_2 := \max_{0 \leq x \leq l} b(x),$$

$$B_i := \{x \in [0, l]; \ b(x) = b_i\} \quad (i = 1, 2).$$

We assume the following assumption.

Assumption 5.1. *The function $a(x)$ is an odd function belonging to $L^\infty(-l, l)$, and it satisfies*

$$\max_{x \in B_1} x < \min_{y \in B_2} y.$$

A typical example of $a(x)$ satisfying Assumption 5.1 is

$$a(x) = \frac{|x|^\alpha}{x} \quad b(x) = \frac{1}{\alpha} |x|^\alpha, \quad (5.2)$$

for $\alpha \geq 1$ and $l > 0$. In case that

$$a(x) = \sin x, \quad b(x) = -\cos x, \quad (5.3)$$

it satisfies Assumption 5.1 if $0 < l < 2\pi$. Another example is

$$a(x) = x^3 - x, \quad b(x) = \frac{1}{4}(x^2 - 1)^2, \quad (5.4)$$

which satisfies Assumption 5.1 if $l > \sqrt{2}$.

In the following arguments, $A(p) \sim B(p)$ as $p \rightarrow \infty$ means that $A(p)/B(p) \rightarrow 1$ as $p \rightarrow \infty$. Now we present the main result of this section.

Theorem 5.2. *Under Assumption 5.1, we have*

$$\lambda_1(p) \sim \left(\int_0^l e^{-pb(x)} dx \right)^{-1} \left(\int_0^l e^{pb(y)} dy \right)^{-1} \quad \text{as } p \rightarrow \infty. \quad (5.5)$$

Proof. We first remark that the principal eigenfunction $u_1(x)$ defined by (5.1) is an even function since $a(x)$ is odd. By means of the change of variables from $x \in [0, l]$ to $r \in [0, 1]$ by $r = \rho(x)$:

$$\rho(x) := \frac{1}{\sigma} \int_0^x e^{pb(y)} dy \quad (0 \leq x \leq l), \quad \sigma := \int_0^l e^{pb(x)} dx,$$

$$U_1(r) := u_1(\rho^{-1}(r)) \quad (0 \leq r \leq 1),$$

it follows that $U_1(r)$ and $\lambda_1(p)$ satisfy

$$\begin{cases} -U_1''(r) = \lambda_1(p) z(r) U_1(r) & (0 < r < 1) \\ U_1'(0) = 0, \quad U_1(1) = 0 \\ U_1(r) > 0 & (0 < r < 1), \end{cases} \quad (5.6)$$

where

$$z(r) := \sigma^2 e^{-2pb(\rho^{-1}(r))} > 0 \quad (0 \leq r \leq 1).$$

We remark that the quantity in the right hand side of (5.5) is given by the following equality

$$\int_0^1 z(r)dr = \left(\int_0^l e^{-pb(x)} dx \right) \left(\int_0^l e^{pb(y)} dy \right),$$

which can be checked by direct calculation.

The eigenvalue problem (5.6) is selfadjoint and it is found that $\lambda_1(p)$ is given by minimum of the Rayleigh quotient:

$$\lambda_1(p) = \frac{\int_0^1 |U_1'(r)|^2 dr}{\int_0^1 z(r)U_1(r)^2 dr} = \min_{\varphi \in V} \frac{\int_0^1 |\varphi'(r)|^2 dr}{\int_0^1 z(r)\varphi(r)^2 dr}, \quad (5.7)$$

where $V := \{\varphi \in H^1(0, 1); \varphi(1) = 0, \varphi \not\equiv 0\}$.

Since

$$U_1'(r) = U_1'(0) + \int_0^r U_1''(s)ds = -\lambda_1(p) \int_0^r z(s)U_1(s)ds < 0 \quad (0 < r \leq 1),$$

we have $\max U_1 = U_1(0)$. Without loss of generality, we can assume that $\max U_1 = U_1(0) = 1$. Among functions $\varphi \in H^1(0, 1)$ with $\varphi(0) = 1$ and $\varphi(1) = 0$, the minimum of the Dirichlet integral $\int_0^1 |\varphi'(r)|^2 dr$ is achieved by $\varphi(r) = 1 - r$. Hence, from the first equality of (5.7), we get an estimate from below:

$$\lambda_1(p) = \frac{\int_0^1 |U_1'(r)|^2 dr}{\int_0^1 z(r)U_1(r)^2 dr} \geq \frac{\int_0^1 |(1-r)'|^2 dr}{\int_0^1 z(r)dr} = \frac{1}{\int_0^1 z(r)dr}. \quad (5.8)$$

On the other hand, from the second equality of (5.7), substituting $\varphi(r) = 1 - r$, we obtain an estimate from above:

$$\lambda_1(p) = \min_{\varphi \in V} \frac{\int_0^1 |\varphi'(r)|^2 dr}{\int_0^1 z(r)\varphi(r)^2 dr} \leq \frac{\int_0^1 |(1-r)'|^2 dr}{\int_0^1 z(r)(1-r)^2 dr} = \frac{1}{\int_0^1 z(r)(1-r)^2 dr}. \quad (5.9)$$

From these two estimates (5.8) and (5.9), if we can show that

$$\frac{\int_0^1 z(r)(1-r)^2 dr}{\int_0^1 z(r)dr} \rightarrow 1 \quad \text{as } p \rightarrow \infty, \quad (5.10)$$

then we obtain the assertion of the theorem.

Let us show (5.10). We define

$$Z_m(p) := \int_0^1 z(r)r^m dr > 0 \quad (m = 0, 1, 2).$$

Then we have

$$\frac{\int_0^1 z(r)(1-r)^2 dr}{\int_0^1 z(r)dr} = 1 - \frac{2Z_1(p) - Z_2(p)}{Z_0(p)}.$$

Since $0 \leq Z_2(p) \leq Z_1(p)$ holds, it is sufficient to show that $Z_1(p)/Z_0(p) \rightarrow 0$ as $p \rightarrow \infty$.

For $m = 0, 1$, we have the following equalities:

$$Z_m(p) = \sigma \int_0^l e^{-pb(x)} \rho(x)^m dx = \iint_{D_m} e^{p(b(y)-b(x))} dx dy,$$

where

$$D_0 := \{(x, y) \mid 0 < x < l, 0 < y < l\}, \quad D_1 := \{(x, y) \mid 0 < y < x < 1\}.$$

We define

$$\omega_m := \max_{(x, y) \in D_m} (b(y) - b(x)) \quad (m = 0, 1).$$

Then, $\omega_0 = b_0$ and Assumption 5.1 is equivalent to $\omega_0 > \omega_1$. Defining

$$G := \left\{ (x, y) \in D_0 \mid b(y) - b(x) \geq \frac{\omega_0 + \omega_1}{2} \right\},$$

we obtain $|G| > 0$ and

$$\frac{Z_1(p)}{Z_0(p)} = \frac{\iint_{D_1} e^{p(b(y)-b(x)-\omega_1)} dx dy}{\iint_{D_0} e^{p(b(y)-b(x)-\omega_1)} dx dy} \leq \frac{\iint_{D_1} dx dy}{\iint_G e^{p(\frac{\omega_0+\omega_1}{2}-\omega_1)} dx dy} = \frac{|D_1|}{|G|} e^{-p\frac{\omega_0-\omega_1}{2}} \rightarrow 0, \quad (5.11)$$

as p tends to infinity. Hence we have proved (5.10). \square

As we have seen in the above proof, the exponential decay phenomenon of the principal eigenvalue occurs under the Assumption 5.1.

Corollary 5.3. *Under Assumption 5.1, we have*

$$\lim_{p \rightarrow \infty} \frac{1}{p} \log \frac{1}{\lambda_1(p)} = b_0. \quad (5.12)$$

In other words, for arbitrary $\varepsilon \in (0, b_0)$, there exists $p_0 > 0$ such that

$$e^{-p(b_0+\varepsilon)} \leq \lambda_1(p) \leq e^{-p(b_0-\varepsilon)} \quad (p \geq p_0). \quad (5.13)$$

Proof. It is sufficient to show (5.12), since (5.13) is equivalent to (5.12). From Theorem 5.2, $\lambda_1(p)Z_0(p) \rightarrow 1$ as $p \rightarrow \infty$ holds. We have (5.12) by the equalities:

$$\lim_{p \rightarrow \infty} \left(\frac{1}{p} \log \frac{1}{\lambda_1(p)} - b_0 \right) = \lim_{p \rightarrow \infty} \left(\frac{1}{p} \log Z_0(p) - b_0 \right) = \lim_{p \rightarrow \infty} \frac{1}{p} \log \left(e^{-pb_0} Z_0(p) \right) = 0, \quad (5.14)$$

where the last equality is shown as follows. From the inequality $e^{-pb_0} Z_0(p) \leq |D_0|$, it follows that,

$$\limsup_{p \rightarrow \infty} \frac{1}{p} \log \left(e^{-pb_0} Z_0(p) \right) \leq \lim_{p \rightarrow \infty} \frac{\log |D_0|}{p} = 0.$$

Moreover, for arbitrary $\varepsilon > 0$, we define $D_0^\varepsilon := \{(x, y) \in D_0; b(y) - b(x) \geq b_0 - \varepsilon\}$. Then, from $|D_0^\varepsilon| > 0$ and the inequality:

$$e^{-pb_0} Z_0(p) = \iint_{D_0} e^{p(b(y)-b(x)-b_0)} dx dy \geq \iint_{D_0^\varepsilon} e^{-p\varepsilon} dx dy = e^{-p\varepsilon} |D_0^\varepsilon|,$$

we obtain

$$\liminf_{p \rightarrow \infty} \frac{1}{p} \log \left(e^{-pb_0} Z_0(p) \right) \geq \lim_{p \rightarrow \infty} \frac{-p\varepsilon + \log |D_0^\varepsilon|}{p} = -\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the last equality of (5.14) follows. \square

For concrete examples of $a(x)$, more precise asymptotic behaviours of $\lambda_1(p)$ can be derived from Theorem 5.2. For example, if $|B_1||B_2| > 0$ then we get

$$\lambda_1(p) \sim \frac{1}{|B_1||B_2|} e^{-pb_0} \quad \text{as } p \rightarrow \infty,$$

since the following estimates hold:

$$\int_0^l e^{-pb(x)} dx \sim |B_1| e^{-pb_1} \quad \text{as } p \rightarrow \infty, \quad \text{if } |B_1| > 0,$$

$$\int_0^l e^{pb(x)} dx \sim |B_2| e^{pb_2} \quad \text{as } p \rightarrow \infty, \quad \text{if } |B_2| > 0.$$

In case that $|B_1||B_2| = 0$, the following lemma is useful.

Lemma 5.4. *Let $L > 0$ and $\mu > 0$. Suppose that $g \in C^0([0, L])$ satisfies $g(x) > 0$ for $x \in (0, L]$ and $\lim_{x \downarrow 0} x^{-\mu} g(x) = 1$. Then we have*

$$\int_0^L e^{-pg(x)} dx \sim \Gamma\left(\frac{1}{\mu} + 1\right) p^{-\frac{1}{\mu}} \quad \text{as } p \rightarrow \infty,$$

where Γ stands for the Gamma function $\Gamma(\zeta) = \int_0^\infty s^{\zeta-1} e^{-s} ds$.

Proof. In case that $g(x) = x^\mu$, by the change of variables $s = px^\mu$, we have

$$\frac{p^{\frac{1}{\mu}}}{\Gamma(1/\mu + 1)} \int_0^L e^{-px^\mu} dx = \frac{p^{\frac{1}{\mu}}}{\Gamma(1/\mu + 1)} \frac{1}{\mu p^{\frac{1}{\mu}}} \int_0^{pL^\mu} s^{\frac{1}{\mu}-1} e^{-s} ds = \frac{1}{\Gamma(1/\mu)} \int_0^{pL^\mu} s^{\frac{1}{\mu}-1} e^{-s} ds \rightarrow 1,$$

as p tends to infinity. Hence, the assertion follows for $g(x) = x^\mu$. For general $g(x)$, we omit a proof but the same asymptotic behaviour can be shown. \square

By using this lemma, for example, we obtain the following proposition.

Proposition 5.5. *For the case (5.2) with $\alpha \geq 1$, we have*

$$\lambda_1(p) \sim \frac{\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha}} l^{\alpha-1}}{\Gamma\left(\frac{1}{\alpha} + 1\right)} p^{\frac{1}{\alpha}+1} e^{-\frac{l^\alpha}{\alpha} p} \quad \text{as } p \rightarrow \infty.$$

In particular, if $a(x) = x$ (i.e., $-u''(x) + pxu'(x) = \lambda u(x)$ ($-l < x < l$)), then

$$\lambda_1(p) \sim \sqrt{\frac{2}{\pi}} l p^{\frac{3}{2}} e^{-\frac{l^2}{2} p} \quad \text{as } p \rightarrow \infty. \quad (5.15)$$

Proof. From Lemma 5.4, we have

$$\int_0^l e^{-pb(x)} dx = \int_0^l e^{-\frac{p}{\alpha} x^\alpha} dx \sim \Gamma\left(\frac{1}{\alpha} + 1\right) \left(\frac{p}{\alpha}\right)^{-\frac{1}{\alpha}} \quad \text{as } p \rightarrow \infty.$$

Using the change of variables $y = l - x$, we also have

$$\int_0^l e^{pb(x)} dx = \int_0^l e^{\frac{p}{\alpha} x^\alpha} dx = e^{\frac{p}{\alpha} l^\alpha} \int_0^l e^{-l^{\alpha-1} p g(y)} dy \sim e^{\frac{p}{\alpha} l^\alpha} \Gamma(2) (l^{\alpha-1} p)^{-1} \quad \text{as } p \rightarrow \infty,$$

where $g(y) := \alpha^{-1}l^{1-\alpha}(l^\alpha - (l-y)^\alpha)$ satisfies the condition of Lemma 5.4 with $\mu = 1$. From Theorem 5.2, we obtain the assertion. \square

Similarly, we can calculate that, for (5.3):

$$\lambda_1(p) \sim \begin{cases} \frac{\sqrt{2}\sin l}{\sqrt{\pi}} p^{\frac{3}{2}} e^{-(1-\cos l)p} & (\text{if } 0 < l < \pi) \\ \frac{1}{2\pi} p e^{-2p} & (\text{if } l = \pi) \\ \frac{1}{\pi} p e^{-2p} & (\text{if } \pi < l < 2\pi) \end{cases} \quad \text{as } p \rightarrow \infty,$$

and for (5.4):

$$\lambda_1(p) \sim \frac{a(l)}{\sqrt{\pi}} p^{\frac{3}{2}} e^{-b(l)p} \quad \text{as } p \rightarrow \infty, \quad \text{if } l > \sqrt{2}.$$

We remark that similar precise asymptotic expansion has been studied in de Groen [4] by means of precise approximation of the eigenfunctions under a different assumption. Our result (5.15) coincides to the result obtained there ((8.11) in [4]).

At the end of this section, we remark that Theorem 5.2 can not be straightforwardly extended to multi-dimensional cases. To see this, we assume that Ω and $a(\mathbf{x})$ satisfy the conditions of Proposition 3.8 with $\mathcal{I}_i = (-l_i, l_i)$ and that each $a_i(x)$ satisfies Assumption 5.1. Then, since $b(\mathbf{x}) = \sum_{i=1}^n b_i(x_i)$ with $b'_i(x_i) = a_i(x_i)$, we have

$$\left(\int_{\Omega} e^{-pb(\mathbf{x})} d\mathbf{x} \right)^{-1} \left(\int_{\Omega} e^{pb(\mathbf{y})} d\mathbf{y} \right)^{-1} \sim 4^{-n} \prod_{i=1}^n \lambda_1^{(i)}(p) \quad \text{as } p \rightarrow \infty,$$

whereas, from Proposition 3.8, we obtain

$$\lambda_1(p) = \sum_{i=1}^n \lambda_1^{(i)}(p).$$

In this case, the asymptotic behaviour of $\lambda_1(p)$ is determined by the slowest decaying $\lambda_1^{(i)}(p)$.

6 Chemotaxis effect in biological colonies

A simple biological interpretation of the exponential decay phenomenon of principal eigenvalue is discussed in this section. We consider the following simple PDE model for aggregation phenomena in biology.

Let $v(\mathbf{x}, t) \geq 0$ be a population density at $\mathbf{x} \in \Omega$ of an organism (e.g. bacteria), which exhibits an aggregation behaviour due to an attractive chemical substance. This property is called chemotaxis and the attractive chemical substance is called chemotaxis substance. We denote by $c(\mathbf{x}, t) \geq 0$ the density of the chemotaxis substance. Then one of the standard PDE description of the chemotaxis effect is given by the following equation.

$$v_t = \Delta v - p \operatorname{div}(v \nabla c) \quad \text{in } \Omega \times (0, \infty), \quad (6.1)$$

where $p > 0$ stands for a chemotaxis parameter. If p is larger, v exhibits stronger chemotaxis.

Keller and Segel ([10], [11]) and other great number of literatures (for example Diaz and Nagai [6], Stevens [16] etc.) have proposed (6.1) nonlinearly coupled with the production of chemotaxis substance by v :

$$\alpha c_t = \Delta c - \gamma c + v \quad \text{in } \Omega \times (0, \infty),$$

with a suitable boundary condition for c , where α and γ are nonnegative constants. There are several results on finite time blow-up of these systems if the initial density $v(\mathbf{x}, 0)$ is large enough. The blowup of the solution in the Keller-Segel model has attracted many mathematicians' interests and its mathematical mechanism has been studied. Please see the above literature and references therein for more details. Especially, a good short review for mathematical results on the Keller-Segel model is found in § 8 of [16].

On the other hand, in [6], under the zero Dirichlet boundary condition for v with $\alpha = 0$, they proved that v decays exponentially to zero as $t \rightarrow \infty$ if $n = 1$ or if $v(\mathbf{x}, 0)$ is sufficiently small for $n \geq 2$. The zero Dirichlet boundary condition for v biologically means the extinction of the organism at the boundary. The result in [6] can be interpreted as that, the colony formed by the chemotaxis aggregation can not sustain eternally against the diffusion and the boundary extinction effect ($v = 0$ on $\partial\Omega$), if the colony size is not large enough.

Let us consider a simpler linear version of this chemotaxis aggregation vs boundary extinction problem. We suppose that the chemotaxis substance c is a priori given and does not change in time, i.e. $c = c(\mathbf{x})$, and we consider the initial-boundary value problem:

$$\begin{cases} v_t = \Delta v - p \operatorname{div}(v \nabla c(\mathbf{x})) & \text{in } \Omega \times (0, \infty) \\ v = 0 & \text{on } \partial\Omega \times (0, \infty) \\ v(\cdot, 0) = v_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \quad (6.2)$$

where $v_0(\mathbf{x}) \geq 0$ is a given initial density function with $v_0 \not\equiv 0$. It is mathematically well-known that the boundary extinction is always stronger than the chemotaxis aggregation. It decays exponentially as

$$v(\mathbf{x}, t) \sim C e^{-\lambda_1(p)t} v_1(\mathbf{x}, p) \quad (t \rightarrow \infty),$$

where $\lambda_1(p)$ is nothing but $\lambda_1(p)$ in Section 4 with $b(\mathbf{x}) = -c(\mathbf{x})$, and $v_1(\mathbf{x}, p)$ is the principal eigenfunction of K_{pb}^* . If $b(\mathbf{x}) = -c(\mathbf{x})$ has the potential well (Definition 4.6) of depth $\omega > 0$, $\lambda_1(p)$ exhibits the exponential decay phenomenon: $\lambda_1(p) \approx O(e^{-\omega p})$ for large p . The potential well condition for $-c(\mathbf{x})$ means that there is a concentration peak of the chemotaxis substance of height ω .

The reciprocal of the principal eigenvalue $1/\lambda_1(p)$ stands for the life span of the colony, since the half-life of exponential decaying quantity $e^{-\lambda t}$ is given by $(\log 2)/\lambda$. Theorem 4.7 is interpreted as that if $c(\mathbf{x})$ has a concentration peak of height $\omega > 0$ then the life span of the colony is exponentially long as

$$\frac{1}{\lambda_1(p)} \approx C e^{\omega p} \quad \text{as } p \rightarrow \infty.$$

In other words, the colony can not sustain eternally, but it sustains in enough long time $O(e^{\omega p})$ if the chemotaxis parameter is sufficiently large.

In Figure 23, a typical profile of colony is shown. This is the eigenfunction v_1 of (3.3) with same $\mathbf{a}(\mathbf{x}) (= -\nabla c(\mathbf{x}))$ as Example 2.2 and $p = 40$. Although the height of this colony is

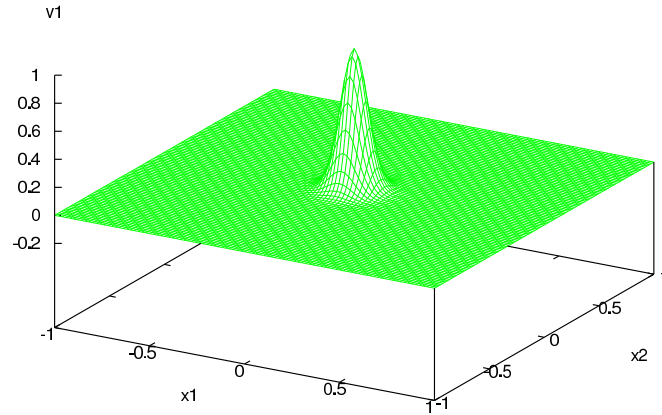


Figure 23: Eigenfunction v_1 of (3.3) corresponding to u_1 of Figure 10.

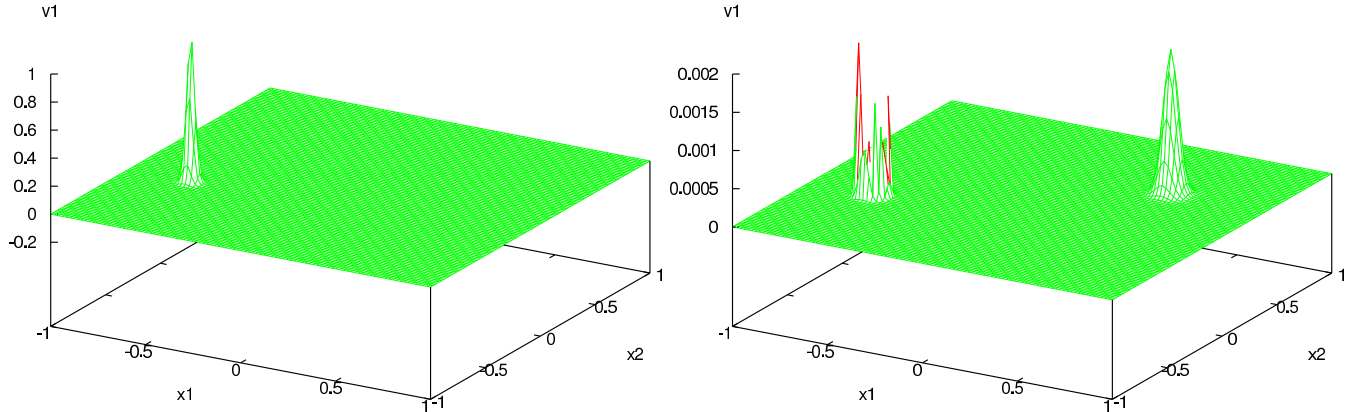


Figure 24: Eigenfunction v_1 of (3.3) corresponding to u_1 of Figure 21 (left), and a scaled figure enlarged in the vertical axis (right).

decreasing in time according to the linear law, but the decay is exponentially slow for large p and it substantially forms a very solid colony. Another profile of v corresponding to Example 2.3 with $p = 100$ is shown in Figure 24, which has two colonies. The height of the right small colony is approximately $1/500$ of the height of the left one. This is interesting in comparison with Figure 21 where the eigenfunction u_1 has two terraces with almost same height.

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